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REGULAR PROGRAMMED MAXIMIN IN DIFFERENTIAL GAMES*

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Conditions are investigated for the coincidence of the programmed maximin and the value of a positional differential game /1-5/. In connection with this the infinitesimal form of the stability property, derived in /6/, is used. It is shown that under the fulfillment of the well-known regularity conditions /3-5/ the programmed maximin function is directionally differentiable and the stability property holds for it. The paper's main results are the necessary and sufficient condition for the coincidence of the programmed maximin and the positional differential game's value when the controlled system's right-hand side is differentiable with respect to the phase variable, and a sufficient regularity condition when the controlled system's right-hand side satisfies a Lipschitz condition in the phase variable. These conditions generalize the previously known /3-5/ regularity conditions. The paper abuts the investigations in /1-10/.

1. Let the motion of a controlled system be described by the equation

$$x^{\cdot} = f(t, x, u, v),\ x \in R^n,\ u \in P \subset R^p,\ v \in Q \subset R^q \tag{1.1}$$
$$f: (-\infty, \theta] \times R^n \times P \times Q \to R^n,\ \|f(t, x, u, v)\| \leqslant \varkappa(1 + \|x\|)$$

Here θ is a fixed instant of time, P and Q are compacta, f is a continuous function satisfying the inequality indicated, where $\|\cdot\|$ is the Euclidean norm, $\varkappa$ is a constant. The payoff is specified by the continuous function $\sigma = \sigma(x(\theta))$ ($x(\cdot)$ is the system's realized motion). We assume at first that the controlled system satisfies the following conditions.

1°. The derivatives $\partial f_i/\partial x_j$ $(i, j = 1, 2, \ldots, n)$ exist and are continuous on $(-\infty, \theta) \times R^n \times P \times Q$.

2°. The equality (the prime denotes transposition)

$$\min_{u \in P} \max_{v \in Q} s'f(t, x, u, v) = \max_{v \in Q} \min_{u \in P} s'f(t, x, u, v)$$
$$\forall\ (t, x) \in (-\infty, \theta] \times R^n,\ s \in R^n$$

is valid.

According to the formalization from /3/, the positional strategies of the first (payoff minimizing) player and the second (payoff maximizing) player are identified with functions defined on the position space $(-\infty, \theta] \times R^n$ and taking values in compacta P and Q, respectively. For any initial position (t_*, x_*) this game has a saddle point and the value $c^\circ(t_*, x_*)$. The function $c(t, x) \to c^\circ(t, x)$ is called the differential game's potential. As was shown in /6/, for a function $c(t, x)$ satisfying a Lipschitz condition to be the differential game's potential, it is necessary and sufficient that it satisfy the boundary condition $c(\theta, x) = \sigma(x)$ and that the inequalities

$$\sup_{v \in Q} \inf_{h \in F(t, x, v)} \partial_- c(t, x)/\partial(1, h) \leqslant 0 \tag{1.2}$$
$$\inf_{u \in P} \sup_{h \in F(t, x, u)} \partial_+ c(t, x)/\partial(1, h) \geqslant 0$$
$$F(t, x, u) = \mathrm{co}\,\{f(t, x, u, v) : v \in Q\}$$
$$F(t, x, v) = \mathrm{co}\,\{f(t, x, u, v) : u \in P\}$$

be fulfilled for all $(t, x) \in (-\infty, \theta) \times R^n$. Here $\partial_- c(t, x)/\partial(1, h)$ and $\partial_+ c(t, x)/\partial(1, h)$ are the lower and upper derivatives of function $c(t, x)$ in the direction $(1, h)$. If the function $c(t, x)$ has a derivative in the direction $(1, h)$ at the point $(t, x) \in (-\infty, \theta) \times R^n$, then

$$\partial_- c(t, x)/\partial(1, h) = \partial_+ c(t, x)/\partial(1, h) = \partial c(t, x)/\partial(1, h)$$

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We introduce into consideration the programmed maximin quantity

$$c^*(t_*, x_*) = \sup_{v(\cdot)\in V} \inf_{u(\cdot)\in U} \sigma(x(\theta, t_*, x_*, u(\cdot), v(\cdot)))$$
$$u(\cdot) : (-\infty, \theta] \mapsto P,\ v(\cdot) : (-\infty, \theta] \mapsto Q$$

where U and V are sets of all measurable functions (programmed controls), $x(\cdot, t_*, x_*, u(\cdot), v(\cdot)) = (x(t), t_* \leqslant t \leqslant \theta)$ is the solution of Eq.(1.1) with $u = u(t)$, $v = v(t)$, satisfying the condition $x(t_*) = x_*$. It is known /3/ that the programmed maximin and the value are related by

$$c^*(t_*, x_*) \leqslant c^\circ(t_*, x_*), \quad \forall (t_*, x_*) \in (-\infty, \theta] \times R^n \tag{1.3}$$

The next statement's validity can also be shown.

Statement. Let the function $c(t, x)$ satisfy the Lipschitz condition, the boundary condition $c(\theta, x) = \sigma(x)$ and the first inequality in (1.2) for all $(t, x) \in (-\infty, \theta) \times R^n$ and let $\mathbf{U}_e$ be a strategy extremal to the set

$$W^{(u)} = \{(t, x) \in [t_*, \theta] \times R^n : c(t, x) \leqslant c(t_*, x_*)\}$$

Then the inequality

$$\sigma(x[\theta]) \leqslant c(t_*, x_*)$$

is fulfilled for any motion $x[t, t_*, x_*, \mathbf{U}_e]$ generated by strategy $\mathbf{U}_e$ from the position (t_*, x_*).

Hence it follows that the quantity $c(t_*, x_*)$ is related to the game's value $c^\circ(t_*, x_*)$ by the inequality

$$c(t_*, x_*) \geqslant c^\circ(t_*, x_*) \tag{1.4}$$

Having taken the programmed maximin function $c^*(t, x)$ as $c(t, x)$ and used inequalities (1.3) and (1.4), we arrive at the next assertion.

Lemma 1.1. For the programmed maximin to coincide everywhere with the game's value, it is necessary and sufficient that the function $c^*(t, x)$ satisfy the first inequality in (1.2) for all $(t, x) \in (-\infty, \theta) \times R^n$.

We remark that when the well-known regularity conditions are fulfilled, the function $c^*(t, x)$ proves to be differentiable in any direction $(1, h)$.

We first look at the linear case. Let

$$f(t, x, u, v) = A(t)x + B(t)u + C(t)v$$

where $A(t)$, $B(t)$ and $C(t)$ are continuous matrix-valued functions of appropriate dimensions. Here the sets P and Q are assumed to be convex compacta. The payoff function $\sigma : R^n \mapsto R$ is convex and satisfies the Lipschitz condition

$$|\sigma(x^{(2)}) - \sigma(x^{(1)})| \leqslant \lambda \|x^{(2)} - x^{(1)}\| \tag{1.5}$$
$$(\lambda = \text{const},\ x^{(i)} \in R^n,\ i = 1, 2)$$

From convex analysis /11/ it is known that the function σ can be taken as

$$\sigma(x) = \max_{l\in L} [l'x - \sigma^*(l)] \tag{1.6}$$
$$\sigma^*(l) = \sup_{x\in R^n} [l'x - \sigma(x)],\ L = \{l \in R^n : \sigma^*(l) < \infty\}$$

Here σ^* is a convex function conjugate to σ, defined by the equality indicated. Set L is bounded on the strength of (1.5). Using (1.6), the Cauchy formula for solving the linear equation, and a minimax theorem /12/ for the function $c^*(t, x)$, we can obtain the expression

$$c^*(t, x) = \max_{l\in L} \Big[l'\Phi(\theta, t)x + \int_t^\theta r_1(\tau, l)\,d\tau + \int_t^\theta r_2(\tau, l)\,d\tau - \sigma^*(l)\Big] \tag{1.7}$$

where $\Phi(t, \tau)$ is the fundamental matrix of solutions of the homogeneous equation

$$x^\cdot = A(t)x,\ r_1(t, l) = \min_{u\in P} [l'\Phi(\theta, t)B(t)u]$$
$$r_2(t, l) = \max_{v\in Q} [l'\Phi(\theta, t)C(t)v]$$

By virtue of the theorem on the directional differentiability of the maximum function /13/, it follows from (1.7) that the function $c^*(t, x)$ has a derivative in any direction $(1, h)$, which is expressed by the formula

$$\frac{\partial c^*(t, x)}{\partial(1, h)} = \max_{l\in L_0(t, x)} [l'\Phi(\theta, t)h - r_1(t, l) - r_2(t, l) - l'\Phi(\theta, t)A(t)x]$$

where $L_0(t, x)$ is the set of vectors from L, on which the maximum in (1.7) is reached. Therefore, by Lemma 1.1, for the value to coincide everywhere with the programmed maximin, it is necessary and sufficient that the inequality

$$\max_{v\in Q}\min_{u\in P}\max_{l\in L_0(t,x)} [l'\Phi(\theta,t)B(t)u + l'\Phi(\theta,t)C(t)v - r_1(t,l) - r_2(t,l)] \leqslant 0 \quad \forall (t,x)\in(-\infty,\theta)\times R^n \tag{1.8}$$

be fulfilled. Inequality (1.8) is a well-known necessary and sufficient regularity condition (see /4/). Here it has been obtained by computing the directional derivative of the programmed maximin function.

Let us now consider the nonlinear case. We assume the fulfillment of the following conditions.

3°. The function $\sigma: R^n \mapsto R$ is continuously differentiable.

4°. The sets

$$V_0(t_*, x_*) = \{v_0(\cdot)\in V : \inf_{u(\cdot)\in U} \sigma(x(\theta, t_*, x_*, u(\cdot), v_0(\cdot))) = c^*(t_*, x_*)\}$$
$$U_0(t_*, x_*, v(\cdot)) = \{u_0(\cdot)\in U : \sigma(x(\theta, t_*, x_*, u_0(\cdot), v(\cdot))) = \inf_{u(\cdot)\in U} \sigma(x(\theta, t_*, x_*, u(\cdot), v(\cdot)))\}$$

are nonempty for any $(t_*, x_*)\in(-\infty,\theta]\times R^n$ and $v(\cdot)\in V$.

5°. For any $(t_*, x_*)\in(-\infty,\theta)\times R^n$ and $v(\cdot)\in V_0(t_*, x_*)$ the set $U_0(t_*, x_*, v(\cdot))$ consists of a single element.

Let $(t_*, x_*, u(\cdot), v(\cdot))\in(-\infty,\theta)\times R^n\times U\times V$. Consider the motion $x(t) = x(t, t_*, x_*, u(\cdot), v(\cdot))$ $(t_*\leqslant t\leqslant\theta)$. By $L(t)$ we denote the derivative (the Jacobi matrix) $\partial f/\partial x$ computed at point $(t, x(t), u(t), v(t))$. We determine the solution $s(\cdot) = (s(t), t_*\leqslant t\leqslant\theta)$ of the adjoint system $s^{\cdot}(t) = -L'(t)s(t)$, satisfying the condition $s(\theta) = \partial\sigma(x(\theta))/\partial x$. The value of function $s(\cdot)$ at point $t = t_*$ is denoted $s[t_*, x_*, u(\cdot), v(\cdot)]$. We note that a unique vector $s[t_*, x_*, u(\cdot), v(\cdot)]$ corresponds to the collection $(t_*, x_*, u(\cdot), v(\cdot))$. Using well-known constructions (/3/,Chapter VI and VII) we can show that the function $c^*(t, x)$ is differentiable in any direction $(1, h)$ at each point $(t_*, x_*)\in(-\infty,\theta)\times R^n$ and that this derivative is expressed

$$\frac{\partial c^*(t_*, x_*)}{\partial(1, h)} = \max_{s\in S(t_*, x_*)} [s'h - \max_{v\in Q}\min_{u\in P} s'f(t_*, x_*, u, v)] \tag{1.9}$$
$$S(t_*, x_*) = \{s[t_*, x_*, u(\cdot), v(\cdot)] : v(\cdot)\in V_0(t_*, x_*),\ u(\cdot)\in U_0(t_*, x_*, v(\cdot))\}$$

Here we omit the derivation of formula (1.9). Lemma 1.1 and formula (1.9) lead us to the following assertion.

Theorem 1.1. Let conditions 1°–5° be fulfilled. Then the programmed maximin function $c^*(t, x)$ is differentiable in any direction $(1, h)$. Further, for the value to coincide everywhere with the programmed maximin, it is necessary and sufficient that the inequality

$$\max_{v\in Q}\ \min_{h\in F(t_*, x_*, v)}\ \max_{s\in S(t_*, x_*)} [s'h - \varkappa(t_*, x_*, s)] \leqslant 0,$$
$$\forall (t_*, x_*)\in(-\infty,\theta)\times R^n,\ \varkappa(t_*, x_*, s) = \max_{v\in Q}\min_{u\in P} s'f(t_*, x_*, u, v)$$

be fulfilled.

The basic result contained in Theorem 1.1 was obtained in /5/. A new proof of this result is proposed in a forthcoming publication.

2. Let us consider a necessary and sufficient condition for the coincidence of the game's value and the programmed maximin when condition 5° is not fulfilled. We introduce some notation. Let Y and Z be metric spaces, $Y_*\subset Y$, $y_*\in\operatorname{cl} Y_*$, and the many-valued mapping $F: Y_*\mapsto 2^Z$ be specified. Following /14/, the symbol $Ls_{y\to y_*}F(y)$ is used to denote the collection of those and only those points z_* for which sequences $y_k\in Y_*$, $z_k\in F(y_k)$ exist such that $y_k\to y_*$, $z_k\to z_*$ as $k\to\infty$. It can be shown that set $Ls_{y\to y_*}F(y)$ is closed. Here and subsequently we assume the fulfillment of conditions 2°–4°. Let $(t_*, x_*)\in(-\infty,\theta)\times R^n$, $h\in R^n$ and $\delta\in(0, \theta - t_*)$. Assume

$$A(t_*, x_*, h, \delta) = \{(u(\cdot), v(\cdot))\in U\times V : c^*(t_* + \delta, x_* + \delta h) - c^*(t_*, x_*) \leqslant \sigma(x(\theta, t_* + \delta, x_* + \delta h, u(\cdot), v(\cdot))) - \sigma(x(\theta, t_*, x_*, u(\cdot), v(\cdot)))\} \tag{2.1}$$

Lemma 2.1. For any $(t_*, x_*) \in (-\infty, \theta) \times R^n, h \in R^n$ and $\delta \in (0, \theta - t_*)$ a pair $(u_*(\cdot), v_*(\cdot)) \in A(t_*, x_*, h, \delta)$ exists such that

$$c^*(t_* + \delta, x_* + \delta h) - c^*(t_*, x_*) = \sigma(x(\theta, t_* + \delta, x_* + \delta h, u_*(\cdot), v_*(\cdot))) - \sigma(x(\theta, t_*, x_*, u_*(\cdot), v_*(\cdot))) \tag{2.2}$$

Proof. We find $v_0(\cdot) \in V_0(t_* + \delta, x_* + \delta h)$ and we select a constant vector $v_* \in Q$. We set $v_1(t) = [v_*$ for $-\infty < t < t_* + \delta$, $v_0(t)$ for $t_* + \delta \leqslant t \leqslant \theta]$. Let $u_1(\cdot) \in U_0(t_*, x_*, v_1(\cdot))$. It can be shown that $(u_1(\cdot), v_1(\cdot)) \in A(t_*, x_*, h, \delta)$. Analogously it can be shown that a pair $(u_2(\cdot), v_2(\cdot)) \in U \times V$ exists for which

$$c^*(t_* + \delta, x_* + \delta h) - c^*(t_*, x_*) \geqslant \sigma(x(\theta, t_* + \delta, x_* + \delta h, u_2(\cdot), v_2(\cdot))) - \sigma(x(\theta, t_*, x_*, u_2(\cdot), v_2(\cdot)))$$

Let $(u_\tau(t), v_\tau(t)) = [(u_1(t), v_1(t)]$ for $-\infty < t < \tau$, $(u_2(t), v_2(t))$ for $\tau \leqslant t \leqslant \theta]$. Consider the function

$$\varphi(\tau) = c^*(t_* + \delta, x_* + \delta h) - c^*(t_*, x_*) - \sigma(x(\theta, t_* + \delta, x_* = \delta h, u_\tau(\cdot), v_\tau(\cdot))) + \sigma(x(\theta, t_*, x_*, u_\tau(\cdot), v_\tau(\cdot))).$$

It is continuous and satisfies the inequalities $\varphi(t_*) \geqslant 0$, $\varphi(\theta) \leqslant 0$. Consequently, there exists $\tau_0 \in [t_*, \theta]$ such that $\varphi(\tau_0) = 0$, i.e., the pair $(u_{\tau_0}(\cdot), v_{\tau_0}(\cdot)) = (u_*(\cdot), v_*(\cdot))$ satisfies equality (2.2).

Let $(u(\cdot), v(\cdot)) \in U \times V$ and the vector $s[t_*, x_*, u(\cdot), v(\cdot)]$ be defined as in Sect.1. We set

$$f_*[t_*, x_*, u(\cdot), v(\cdot), \delta] = \delta^{-1} \int_{t_*}^{t_*+\delta} f(t_*, x_*, u(t), v(t))\, dt \tag{2.3}$$

$$B(t_*, x_*, h, \delta) = \{(s[t_*, x_*, u(\cdot), v(\cdot)], f_*[t_*, x_*, u(\cdot), v(\cdot), \delta]) \in R^n \times R^n : (u(\cdot), v(\cdot)) \in A(t_*, x_*, h, \delta)\}$$

$$B_0(t_*, x_*, h) = Ls_{\delta\to+0} B(t_*, x_*, h, \delta)$$

$$B_0(t_*, x_*, h) \neq \phi, \quad \forall (t_*, x_*, h) \in (-\infty, \theta) R^n \times R^n$$

Theorem 2.1. Let conditions $1^\circ - 4^\circ$ be fulfilled. For the value $c^\circ(t_*, x_*)$ to coincide everywhere with the programmed maximin $c^*(t_*, x_*)$, it is necessary and sufficient that the inequality

$$\sup_v \inf_h \min_{(s, f_*)} s'(h - f_*) \leqslant 0, \quad \forall (t_*, x_*) \in (-\infty, \theta) \times R^n \tag{2.4}$$
$$v \in Q,\ h \in F(t_*, x_*, v),\ (s, f_*) \in B_0(t_*, x_*, h)$$

be fulfilled.

Proof. Sufficiency. Suppose that inequality (2.4) is satisfied. Then for every $v \in Q$ and $\varepsilon > 0$ there exist $h_* \in F(t_*, x_*, v)$ and $(s_*, f_*) \in B_0(t_*, x_*, h_*)$ such that

$$s_*'(h_* - f_*) \leqslant \varepsilon \tag{2.5}$$

By the definition of $B_0(t_*, x_*, h_*)$ we can choose $\delta_k \to +0$, $(s_k, f_k) \in B(t_*, x_*, h_*, \delta_k)$, $(u_k(\cdot), v_k(\cdot)) \in A(t_*, x_*, h_*, \delta_k)$ such that $(s_k, f_k) \to (s_*, f_*)$ and

$$s_k = s[t_*, x_*, u_k(\cdot), v_k(\cdot)],\ f_k = \delta_k^{-1} \int_{t_*}^{t_*+\delta_k} f(t_*, x_*, u_k(t), v_k(t))\, dt \tag{2.6}$$

$$c^*(t_* + \delta_k, x_* + \delta_k h_*) - c^*(t_*, x_*) \leqslant \sigma(x(\theta, t_* + \delta_k, x_* + \delta_k h_*, u_k(\cdot), v_k(\cdot))) - \sigma(x(\theta, t_*, x_*, u_k(\cdot), v_k(\cdot))) \tag{2.7}$$

Using variational equations, the right-hand side of (2.7) can be estimated by the quantity $s_k'(h_* - f_k)\delta_k + o(\delta_k)$. Because of (2.5)

$$\frac{\partial_- c^*(t_*, x_*)}{\partial(1, h_*)} \leqslant \varliminf_{k\to\infty} \delta_k^{-1} [c^*(t_* + \delta_k, x_* + \delta_k h_*) - c^*(t_*, x_*)] \leqslant \lim_{k\to\infty} s_k'(h_* - f_k) = s_*'(h_* - f_*) \leqslant \varepsilon$$

Since $\varepsilon > 0$ and $v \in Q$ are arbitrary and $h_* \in F(t_*, x_*, v)$, by Lemma 1.1 the value coincides everywhere with the programmed maximin.

Necessity. By Lemma 1.1, for every $v \in Q$ and $\varepsilon > 0$ there exist $h_* \in F(t_*, x_*, v)$, $\delta_k \to +0$ such that

$$c^*(t_* + \delta_k, x_* + \delta_k h_*) - c^*(t_*, x_*) \leqslant \varepsilon\delta_k \tag{2.8}$$

On the strength of Lemma 2.1 we can select a pair $(u_k(\cdot), v_k(\cdot)) \in A(t_*, x_*, h_*, \delta_*)$ satisfying (2.2). Because of (2.8) we obtain

$$\sigma(x(\theta,\ t_* + \delta_k,\ x_* + \delta_k h_*,\ u_k(\cdot), v_k(\cdot))) - \sigma(x(\theta,\ t_*, x_*, u_k(\cdot),\ v_k(\cdot))) \leqslant \varepsilon\,\delta_k$$

The left-hand side of (2.8) equals $s_k'(h_* - f_k)\,\delta_k + o(\delta_k)$, where s_k and f_k are defined by (2.6). Therefore

$$s_k'(h_* - f_k)\,\delta_k \leqslant 0(\delta_k) + \varepsilon\delta_k$$

Consequently

$$\min_{(s,\, f_*)\in B_0(t_*,\, x_*,\, h_*)} s'(h_* - f_*) \leqslant \varepsilon$$

and, hence, (2.4) is fulfilled. The theorem is proved.

If condition 2° is not fulfilled, then the positional differential game has an equilibrium situation in other classes of positional strategies of both players, and a corresponding programmed construction and a corresponding definition of programmed maximin are known for each of these types of differential games /3,5/. Using these definitions, we can construct the set B_0 and obtain analogs of Theorems 1.1 and 2.1. We remark as well that Theorems 1.1 and 2.1 remain valid, with appropriate modifications, in the case the function σ is not differentiable, but is given as

$$\sigma(x) = \min_{m\in M} \varphi(x, m)$$

where M is a compactum, the function φ is continuous in all variables and is continuously differentiable in x, as well as in the case condition 4° is not fulfilled.

3. We consider the case when function f satisfies condition 2° and the one following:

$$6^\circ.\ f(t, x, u, v) = f_1(t, x) + f_2(t, u, v),\ (t, x, u, v) \in (-\infty, \theta] \times R^n \times P \times Q$$

$$\| f_1(t, x^{(2)}) - f_1(t, x^{(1)}) \| \leqslant \lambda(G)\, \| x^{(2)} - x^{(1)} \|, \quad (t, x^{(i)}) \in G, \quad (i = 1, 2)$$

where G is any bounded region in $(-\infty, \theta] \times R^n$, $\lambda(G)$ is the Lipschitz constant. Following /15/, we consider the many-valued mapping $(t, x, P) \to K(t, x, p)$, where for any $(t, x, p) \in (-\infty, \theta] \times R^n \times R^n$ we set

$$K(t, x, p) = \operatorname*{Ls}_{\delta\to+0,\ y\to x} \{[f_1(t, y + \delta p) - f_1(t, y)]\,\delta^{-1}\} \tag{3.1}$$

We note that here, in contrast to /15/ wherein $y = x$ in definition (3.1), we require that point y vary in a neighborhood of point x.

Let $x(\cdot): [t_*, \theta] \to R^n$ be a continuous function and $p_* \in R^n$. By the symbol $P(t_*, p_*, x(\cdot))$ we denote the collection of points $p^* \in R^n$ for each of which a solution $p(\cdot): [t_*, \theta] \to R^n$ of the differential inclusion

$$p^\cdot(t) \in K(t,\ x(t),\ p(t)) \tag{3.2}$$

exists satisfying the conditions $p(t_*) = p_*$, $p(\theta) = p^*$. We note that $P(t_*, p_*, x(\cdot)) \neq \varnothing$. Assume that $(t_*, x_*) \in (-\infty, \theta) \times R^n$, $\delta_k \to +0$, $(u_k(\cdot), v_k(\cdot)) \in U \times V$ and $h \in R^n$ have been chosen. Consider the equation

$$x^\cdot(t) = f_1(t,\ x(t)) + f_2(t.\ u_k(t),\ v_k(t)) \tag{3.3}$$

A solution of Eq. (3.3), satisfying the condition $x(t_*) = x_*$, is denoted $x_k(t)$ $(t_* \leqslant t \leqslant \theta)$. By $y_k(t)$ $(t_* \leqslant t \leqslant \theta)$ we denote the solution of the same Eq. (3.3), satisfying the condition $y_k(t_* + \delta_k) = x_* + \delta_k h + o(\delta_k)$. Let

$$\lim_{k\to\infty} \delta_k^{-1} \int_{t_*}^{t_* + \delta_k} f(t_*, x_*, u_k(t), v_k(t))\, dt = f_*, \quad \lim_{k\to\infty} x_k(t) = x(t)$$
$$\forall t \in [t_*,\ \theta],\ \Delta x_k(t) = y_k(t) - x_k(t)$$

Lemma 3.1. All partial limits of the sequence $\Delta x_k(\theta)\delta_k^{-1}$ are contained in set $P(t_*, h - f_*, x(\cdot))$. Lemma 3.1 is proved on the basis of a construction in /15/. We set

$$C(t_*, x_*, h, \delta) = \{(x(\cdot, t_*, x_*, u(\cdot), v(\cdot)),\ f_*[t_*, x_*, u(\cdot), v(\cdot), \delta]):\quad (u(\cdot), v(\cdot)) \in A(t_*, x_*, h, \delta)\}$$
$$C_0(t_*, x_*, h) = \operatorname*{Ls}_{\delta\to+0} C(t_*, x_*, h, \delta)$$

where the vector f_* and the set A have been defined by (2.4) and (2.1).

Theorem 3.1. Let conditions $2^\circ - 4^\circ$ and 6° be fulfilled. For the value of the positional differential game to coincide everywhere with the programmed maximin, it is sufficient that the inequality

$$\sup_{v}\ \inf_{(h,x(\cdot),f_*)}\ \max_{p}\left[\frac{\partial\sigma(x(\theta))}{\partial x}\right]' p \leqslant 0 \tag{3.4}$$
$$\forall\ (t_*,\ x_*) \in (-\infty,\ \theta)\times R^n$$
$$v\in Q,\ h\in F(t_*,\ x_*,\ v),\ (x(\cdot),\ f_*)\in C_0(t_*,\ x_*,\ n)$$
$$p\in P(t_*,\ h-f_*,\ x(\cdot))$$

be fulfilled.

Theorem 3.1 is proved by the scheme used to prove Theorem 2.1, using Lemma 3.1. We remark that if condition 1° is fulfilled, then inequality (3.4) is equivalent to inequality (2.4). Let us assume that system (1.1) satisfies condition 5° in addition to conditions 2°—4° and 6°. We select a vector $v\in Q$ and we define the sets

$$V[t_*,x_*,v,h,\delta]=\{v_0(\cdot)\in V_0(t_*+\delta,x_*+\delta h): v_0(t)=v \text{ for } -\infty< t< t_*+\delta\}$$

$$F[t_*,\ x_*,\ v,\ h,\ \delta]=\{f(t_*,\ x_*,\ u(\cdot),\ v(\cdot),\ \delta]:\ v(\cdot)\in V[t_*,\ x_*,\ v,\ h,\ \delta],\ u(\cdot)\in U_0(t_*,\ x_*,\ v(\cdot))\}$$
$$F_0[t_*,x_*,v,h]=\operatorname*{Ls}_{\delta\to+0} F[t_*,x_*,v,h,\delta]$$

$$X(t_*,\ x_*)=\{x(\cdot,\ t_*,\ x_*,u(\cdot),\ v(\cdot)):\ v(\cdot)\in V_0(t_*,\ x_*),\ u(\cdot)\in U_0(t_*,\ x_*.\ v(\cdot))\}$$

Theorem 3.2. Let conditions 2°—6° be fulfilled. For the value to coincide everywhere with the programmed maximin, it is sufficient that the inequality

$$\sup_{v}\ \inf_{(h,f)}\ \sup_{(x(\cdot),p)}\left[\frac{\partial\sigma(x(\theta))}{\partial x}\right]' p \leqslant 0 \tag{3.5}$$
$$\forall\ (t_*,\ x_*)\in(-\infty,\ \theta)\times R^n$$
$$v\in Q,\ h\in F(t_*,\ x_*,\ v),\ f\in F_0[t_*,\ x_*,\ v,\ h]$$
$$x(\cdot)\in X(t_*,\ x_*),\ p\in P(t_*,\ h-f,\ x(\cdot))$$

be fulfilled.

Theorem 3.2 is a corollary of Theorem 3.1, since inequality (3.4) follows from (3.5).

Example. Let the motions of two controlled objects be described by the equations

$$x^{\cdot\cdot}=-k(x^\cdot)x^\cdot+u,\ y^{\cdot\cdot}=-\gamma y^\cdot+v \tag{3.6}$$
$$k(x^\cdot)=\alpha,\ x^\cdot\geqslant 0;\ k(x^\cdot)=\beta,\ x^\cdot<0;\quad \alpha>\beta>0,\ \gamma>0,\quad |u|\leqslant\lambda_1$$
$$|v|\leqslant\lambda_2$$

Assume that the game termination instant θ has been fixed. The quantity $(x(\theta)-y(\theta))^2$ is the payoff. The right-hand side of system (3.6) is not differentiable with respect to the phase variable, but does satisfy condition 6°. It can be shown that when the conditions

$$\lambda_1\geqslant\lambda_2,\ \lambda_1/\alpha\geqslant\lambda_2/\gamma \tag{3.7}$$

are fulfilled, Theorem 3.2 is applicable to this problem, and, consequently, the game's value coincides everywhere with the programmed maximin. We note that inequalities of form (2.7) were first obtained in /7/, where a pursuit problem for system (3.6) with $\alpha=\beta$ was analyzed.

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